

$$\begin{aligned}
&> \text{restart;} \\
&> u := \sin^2(\text{Pi} \cdot (x_1 + t \cdot (x_2 - x_1))) \cdot \sin(2 \cdot \text{Pi} \cdot (y_1 + t \cdot (y_2 - y_1))); \\
&\quad u := \sin(\pi (x_1 + t (x_2 - x_1)))^2 \sin(2 \pi (y_1 + t (y_2 - y_1))) \tag{1}
\end{aligned}$$

$$\begin{aligned}
&> fu := \text{unapply}(u, t, x_1, x_2, y_1, y_2); \\
fu &:= (t, x_1, x_2, y_1, y_2) \rightarrow \sin(\pi (x_1 + t (x_2 - x_1)))^2 \sin(2 \pi (y_1 + t (y_2 - y_1))) \tag{2}
\end{aligned}$$

$$\begin{aligned}
&> Fu := \text{int}(fu(t, x_1, x_2, y_1, y_2), t); \\
Fu &:= \frac{1}{2} \frac{\cos((-2 \pi y_2 + 2 \pi y_1) t - 2 \pi y_1)}{-2 \pi y_2 + 2 \pi y_1} \\
&\quad - \frac{1}{4} \frac{\cos((-2 \pi y_2 + 2 \pi y_1 - 2 \pi x_2 + 2 \pi x_1) t - 2 \pi y_1 - 2 \pi x_1)}{-2 \pi y_2 + 2 \pi y_1 - 2 \pi x_2 + 2 \pi x_1} \\
&\quad + \frac{1}{4} \frac{\cos((2 \pi y_2 - 2 \pi y_1 - 2 \pi x_2 + 2 \pi x_1) t + 2 \pi y_1 - 2 \pi x_1)}{2 \pi y_2 - 2 \pi y_1 - 2 \pi x_2 + 2 \pi x_1} \tag{3}
\end{aligned}$$

$$\begin{aligned}
&> \\
&> v := -\sin^2(\text{Pi} \cdot (y_1 + t \cdot (y_2 - y_1))) \cdot \sin(2 \cdot \text{Pi} \cdot (x_1 + t \cdot (x_2 - x_1))); \\
&\quad v := -\sin(\pi (y_1 + t (y_2 - y_1)))^2 \sin(2 \pi (x_1 + t (x_2 - x_1))) \tag{4}
\end{aligned}$$

$$\begin{aligned}
&> fv := \text{unapply}(v, t, x_1, x_2, y_1, y_2); \\
fv &:= (t, x_1, x_2, y_1, y_2) \rightarrow -\sin(\pi (y_1 + t (y_2 - y_1)))^2 \sin(2 \pi (x_1 + t (x_2 - x_1))) \tag{5}
\end{aligned}$$

$$\begin{aligned}
&> Fv := \text{int}(fv(t, x_1, x_2, y_1, y_2), t); \\
Fv &:= -\frac{1}{2} \frac{\cos((-2 \pi x_2 + 2 \pi x_1) t - 2 \pi x_1)}{-2 \pi x_2 + 2 \pi x_1} \\
&\quad + \frac{1}{4} \frac{\cos((-2 \pi y_2 + 2 \pi y_1 - 2 \pi x_2 + 2 \pi x_1) t - 2 \pi y_1 - 2 \pi x_1)}{-2 \pi y_2 + 2 \pi y_1 - 2 \pi x_2 + 2 \pi x_1} \\
&\quad + \frac{1}{4} \frac{\cos((2 \pi y_2 - 2 \pi y_1 - 2 \pi x_2 + 2 \pi x_1) t + 2 \pi y_1 - 2 \pi x_1)}{2 \pi y_2 - 2 \pi y_1 - 2 \pi x_2 + 2 \pi x_1} \tag{6}
\end{aligned}$$

$$\begin{aligned}
&> \\
&\frac{1}{2} \frac{\cos(2 \pi y_2)}{-2 \pi y_2 + 2 \pi y_1} - \frac{1}{4} \frac{\cos(2 \pi y_2 + 2 \pi x_2)}{-2 \pi y_2 + 2 \pi y_1 - 2 \pi x_2 + 2 \pi x_1} \\
&\quad + \frac{1}{4} \frac{\cos(-2 \pi y_2 + 2 \pi x_2)}{2 \pi y_2 - 2 \pi y_1 - 2 \pi x_2 + 2 \pi x_1} \tag{7}
\end{aligned}$$

$$\begin{aligned}
&> Fut := \text{unapply}(Fu, t); \\
Fut &:= t \rightarrow \frac{1}{2} \frac{\cos((-2 \pi y_2 + 2 \pi y_1) t - 2 \pi y_1)}{-2 \pi y_2 + 2 \pi y_1} \tag{8}
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{4} \frac{\cos\left(\left(-2\pi y_2 + 2\pi y_1 - 2\pi x_2 + 2\pi x_1\right)t - 2\pi y_1 - 2\pi x_1\right)}{-2\pi y_2 + 2\pi y_1 - 2\pi x_2 + 2\pi x_1} \\
& + \frac{1}{4} \frac{\cos\left(\left(2\pi y_2 - 2\pi y_1 - 2\pi x_2 + 2\pi x_1\right)t + 2\pi y_1 - 2\pi x_1\right)}{2\pi y_2 - 2\pi y_1 - 2\pi x_2 + 2\pi x_1}
\end{aligned}$$

> $Fvt := unapply(Fv, t);$

$$\begin{aligned}
Fvt := t \rightarrow & - \frac{1}{2} \frac{\cos\left(\left(-2\pi x_2 + 2\pi x_1\right)t - 2\pi x_1\right)}{-2\pi x_2 + 2\pi x_1} \\
& + \frac{1}{4} \frac{\cos\left(\left(-2\pi y_2 + 2\pi y_1 - 2\pi x_2 + 2\pi x_1\right)t - 2\pi y_1 - 2\pi x_1\right)}{-2\pi y_2 + 2\pi y_1 - 2\pi x_2 + 2\pi x_1} \\
& + \frac{1}{4} \frac{\cos\left(\left(2\pi y_2 - 2\pi y_1 - 2\pi x_2 + 2\pi x_1\right)t + 2\pi y_1 - 2\pi x_1\right)}{2\pi y_2 - 2\pi y_1 - 2\pi x_2 + 2\pi x_1}
\end{aligned} \tag{9}$$

> $avgu := unapply(Fut(1) - Fut(0), x_1, x_2, y_1, y_2);$

$$\begin{aligned}
avgu := (x_1, x_2, y_1, y_2) \rightarrow & \frac{1}{2} \frac{\cos(2\pi y_2)}{-2\pi y_2 + 2\pi y_1} \\
& - \frac{1}{4} \frac{\cos(2\pi y_2 + 2\pi x_2)}{-2\pi y_2 + 2\pi y_1 - 2\pi x_2 + 2\pi x_1} \\
& + \frac{1}{4} \frac{\cos(-2\pi y_2 + 2\pi x_2)}{2\pi y_2 - 2\pi y_1 - 2\pi x_2 + 2\pi x_1} - \frac{1}{2} \frac{\cos(2\pi y_1)}{-2\pi y_2 + 2\pi y_1} \\
& + \frac{1}{4} \frac{\cos(2\pi y_1 + 2\pi x_1)}{-2\pi y_2 + 2\pi y_1 - 2\pi x_2 + 2\pi x_1} \\
& - \frac{1}{4} \frac{\cos(-2\pi y_1 + 2\pi x_1)}{2\pi y_2 - 2\pi y_1 - 2\pi x_2 + 2\pi x_1}
\end{aligned} \tag{10}$$

> $avgv := unapply(Fvt(1) - Fvt(0), x_1, x_2, y_1, y_2);$

$$\begin{aligned}
avgv := (x_1, x_2, y_1, y_2) \rightarrow & - \frac{1}{2} \frac{\cos(2\pi x_2)}{-2\pi x_2 + 2\pi x_1} \\
& + \frac{1}{4} \frac{\cos(2\pi y_2 + 2\pi x_2)}{-2\pi y_2 + 2\pi y_1 - 2\pi x_2 + 2\pi x_1} \\
& + \frac{1}{4} \frac{\cos(-2\pi y_2 + 2\pi x_2)}{2\pi y_2 - 2\pi y_1 - 2\pi x_2 + 2\pi x_1} + \frac{1}{2} \frac{\cos(2\pi x_1)}{-2\pi x_2 + 2\pi x_1} \\
& - \frac{1}{4} \frac{\cos(2\pi y_1 + 2\pi x_1)}{-2\pi y_2 + 2\pi y_1 - 2\pi x_2 + 2\pi x_1} \\
& - \frac{1}{4} \frac{\cos(-2\pi y_1 + 2\pi x_1)}{2\pi y_2 - 2\pi y_1 - 2\pi x_2 + 2\pi x_1}
\end{aligned} \tag{11}$$

> $avgu\left(\frac{1}{2\text{Pi}} \cdot x_1, \frac{1}{2\text{Pi}} \cdot x_2, \frac{1}{2\text{Pi}} \cdot y_1, \frac{1}{2\text{Pi}} \cdot y_2\right);$

$$\left[\begin{aligned} & \frac{1}{2} \frac{\cos(y_2)}{-y_2 + y_1} - \frac{1}{4} \frac{\cos(y_2 + x_2)}{-y_2 + y_1 - x_2 + x_1} + \frac{1}{4} \frac{\cos(-y_2 + x_2)}{y_2 - y_1 - x_2 + x_1} - \frac{1}{2} \frac{\cos(y_1)}{-y_2 + y_1} \\ & + \frac{1}{4} \frac{\cos(y_1 + x_1)}{-y_2 + y_1 - x_2 + x_1} - \frac{1}{4} \frac{\cos(-y_1 + x_1)}{y_2 - y_1 - x_2 + x_1} \end{aligned} \right] \quad (12)$$

$$\begin{aligned} & \rightarrow \text{avgv} \left(\frac{1}{2 \text{ Pi}} \cdot x_1, \frac{1}{2 \text{ Pi}} \cdot x_2, \frac{1}{2 \text{ Pi}} \cdot y_1, \frac{1}{2 \text{ Pi}} \cdot y_2 \right); \\ & - \frac{1}{2} \frac{\cos(x_2)}{-x_2 + x_1} + \frac{1}{4} \frac{\cos(y_2 + x_2)}{-y_2 + y_1 - x_2 + x_1} + \frac{1}{4} \frac{\cos(-y_2 + x_2)}{y_2 - y_1 - x_2 + x_1} + \frac{1}{2} \frac{\cos(x_1)}{-x_2 + x_1} \\ & - \frac{1}{4} \frac{\cos(y_1 + x_1)}{-y_2 + y_1 - x_2 + x_1} - \frac{1}{4} \frac{\cos(-y_1 + x_1)}{y_2 - y_1 - x_2 + x_1} \end{aligned} \quad (13)$$